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Semi-classical resolvent estimates for the Schrödinger operator on non-compact complete Riemannian manifolds

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Abstract. We prove uniform semi-classical estimates for the resolvent of the Schrödinger operator $h^2\Delta_g + V(x)$, $0 < h \ll 1$, at a nontrapping energy level $E > 0$, where V is a real-valued non-negative potential and Δ_g denotes the positive Laplace-Beltrami operator on a non-compact complete Riemannian manifold which may have a nonempty compact smooth boundary.

Keywords: semi-classical resolvent estimates, non-trapping energy level, generalized geodesics.

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1 Introduction and statement of results

The purpose of this note is to obtain semi-classical resolvent estimates for the operator $h^2\Delta_g + V(x)$ at a nontrapping energy level $E > 0$ on a large class of non-compact complete Riemannian manifolds, (M, g), $\dim M = n \geq 2$, (which may have a compact boundary ∂M of class C^∞) and for a large class of non-negative potentials $V \in C^\infty(\overline{M})$, where Δ_g denotes the positive Laplace-Beltrami operator on (M, g), and $h > 0$ is a small parameter. The manifolds we are going to consider are of the form $M = X_0 \cup X$, where X_0 is a compact, connected Riemannian manifold with a metric $g_{|X_0}$ of class $C^\infty(\overline{X}_0)$ with a compact boundary $\partial X_0 = \partial M \cup \partial X$, $\partial M \cap \partial X = \emptyset$, $X = [r_0, +\infty) \times S$, $r_0 \gg 1$, with metric $g_{|X} := dr^2 + \sigma(r)$. Here $(S, \sigma(r))$ is an $n-1$ dimensional compact Riemannian manifold without boundary equipped with a family of Riemannian metrics $\sigma(r)$ depending smoothly on r which can be written in

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any local coordinates $\theta \in S$ in the form

$$\sigma(r) = \sum_{i,j} g_{ij}(r,\theta)d\theta_i d\theta_j, \quad g_{ij} \in C^\infty(X).$$

Denote $X_r = [r, +\infty) \times S$. Clearly, ∂X_r can be identified with the Riemannian manifold $(S, \sigma(r))$ with the Laplace-Beltrami operator $\Delta_{\partial X_r}$ written as follows

$$\Delta_{\partial X_r} = -p^{-1} \sum_{i,j} \partial_{\theta_i}(p g^{ij} \partial_{\theta_j}),$$

where (g^{ij}) is the inverse matrix to (g_{ij}) and $p = (\det(g_{ij}))^{1/2} = (\det(g^{ij}))^{-1/2}$. We have

$$\Delta_X := \Delta_g|_X = -p^{-1}\partial_r(p\partial_r) + \Delta_{\partial X_r} = -\partial_r^2 - \frac{p'}{p}\partial_r + \Delta_{\partial X_r},$$

where $p' = \partial p/\partial r$. We have the identity

$$p^{1/2}\Delta_X p^{-1/2} = -\partial_r^2 + \Lambda_r + q(r,\theta), \tag{1.1}$$

where

$$\Lambda_r = -\sum_{i,j} \partial_{\theta_i}(g^{ij}\partial_{\theta_j}),$$

and q is an effective potential given by

$$q(r,\theta) = (2p)^{-2}\left(\frac{\partial p}{\partial r}\right)^2 + (2p)^{-2}\sum_{i,j}\frac{\partial p}{\partial \theta_i}\frac{\partial p}{\partial \theta_j}g^{ij} + 2^{-1}p\Delta_X(p^{-1}).$$

We suppose that $q = q_1 + q_2$, where q_1 and q_2 are real-valued functions satisfying

$$|q_1(r,\theta)| \le C, \quad \frac{\partial q_1}{\partial r}(r,\theta) \le Cr^{-1-\delta_0}, \quad |q_2(r,\theta)| \le Cr^{-1-\delta_0}, \tag{1.2}$$

with constants $C, \delta_0 > 0$. Denote

$$h(r,\theta,\xi) = \sum_{i,j} g^{ij}(r,\theta)\xi_i\xi_j, \quad (\theta,\xi) \in T^*S.$$

We also suppose that

$$-\frac{\partial h}{\partial r}(r,\theta,\xi) \ge \frac{C}{r}h(r,\theta,\xi), \quad \forall(\theta,\xi) \in T^*S, \tag{1.3}$$

with a constant $C > 0$. Note that this class of manifolds has already been considered in [3], [12].

Let $V \in C^\infty(\overline{M})$ be a real-valued function, $V(x) \geq 0$, $\forall x \in M$, such that $V(r, \theta) := V|_X$ satisfies

$$|V(r,\theta)| \leq C_1, \quad \frac{\partial V}{\partial r}(r,\theta) \leq C_1 r^{-1-\delta_1}, \tag{1.4}$$

with constants $C_1, \delta_1 > 0$.

Given $0 < h \ll 1$, denote by $G(h)$ the selfadjoint realization of the Schrödinger operator $h^2\Delta_g + V(x)$ on the Hilbert space $H = L^2(M, d\text{Vol}_g)$ with Dirichlet or Neumann boundary conditions, $Bu = 0$, on ∂M. Fix an energy level $E > 0$ such that

$$E - V(x) \geq C_2, \quad \forall x \in M, \tag{1.5}$$

with a constant $C_2 > 0$. Let $h_0(x,\xi)$, $(x,\xi) \in T^*M$, denote the principal symbol of Δ_g, and set

$$p_E(x,\xi) = (E - V(x))^{-1} h_0(x,\xi).$$

The energy level $E > 0$ satisfying (1.5) will be said to be non-trapping for the operator $G(h)$ if for $\forall a \geq r_0$, $\exists T = T(a) > 0$ so that for every generalized geodesics, $\gamma(t)$, associated to the Hamiltonian $p_E(x,\xi)$, with $\gamma(0) \in M \setminus X_a$, there exists $0 < \tau \leq T$ with $\gamma(\tau) \in X_a$. Recall that a generalized geodesics (associated to p_E) is the projection on M of the generalized bicharactersitcs associated to the Hamiltonian p_E (see [7], [8] for the precise definition).

Given a real s, choose a real-valued function $\chi_s \in C^\infty(\overline{M})$, $\chi_s = 1$ on $M \setminus X_{r_0+1/2}$, $\chi_s|_X$ depending only on r, $\chi_s = r^{-s}$ on X_{r_0+1}. Our main result is the following

Theorem 1.1. *Assume(1.2)-(1.4) fulfilled. If $E > 0$ satisfying (1.5) is a non-trapping energy level, then for every $s > 1/2$, there exist constants $C, h_0 > 0$, so that for $0 < h \leq h_0$, $0 < \varepsilon \leq 1$, the following estimate holds*

$$\|\chi_s(G(h) - E \pm i\varepsilon)^{-1}\chi_s\|_{\mathcal{L}(H)} \leq Ch^{-1}. \tag{1.6}$$

Remark 1. When $V \equiv 0$ the estimate (1.6) is equivalent to the high frequency resolvent estimate proved in [12] (see Theorem 1.1).

Remark 2. Using Proposition 2.3 below instead of Proposition 2.4 of [3], one can show in the same way as in [3] that we have an analogue of (1.6) without the non-trapping assumption but with $O\left(e^{C/h}\right)$, $C > 0$, in the RHS. Such an exponential bound for the resolvent has been first obtained by Burq [1] for a class of long-range perturbations of the Euclidean Laplacian.

The estimate (1.6) has been first proved in the case of the operator $h^2\Delta + V(x)$ on $\mathbf{R}^n$, $\Delta = -\sum_{j=1}^n \partial_{x_j}^2$ being the Euclidean Laplacian and V a long-range potential (see [5], [6], [10]), and then extended to more general perturbations on $\mathbf{R}^n$ (see [4], [9]). In all these papers the proof was based on Mourre's commutator method. Vasy and Zworski [11] proved (1.6) in the case of asymptotically Euclidean manifolds without using Mourre's method. However, their proof has been still based on what is an essential ingredient in Mourre's method, namely the existence of a global escape function due to the non-trapping condition. We would like to emphasise on the fact that such a global escape function cannot be constructed when the boundary ∂M is not empty. Note also that the manifold studied in [11] is isometric to a manifold, (M, g), with $\partial M = \emptyset$, belonging to the class described above. Let us also mention the work [2] where an estimate like (1.6) for the cutoff resolvent in a strip was proved in the case of compactly supported perturbations of the Euclidean Laplacian.

Our approach is quite different from those developed in the papers mentioned above. We use Melrose-Sjöstrand [7], [8] results on propagation of C^∞ singularities to get an uniform semi-classical estimate on $M \setminus X_a$, $\forall a \geq r_0$ (see Proposition 2.1). Then we combine this estimate with an estimate on X_b, $b >> r_0$ (see Proposition 2.3), which is a generalization of an estimate already proved in [3] (see Proposition 2.4) in the case of $V \equiv 0$. To our best knowledge, it is the first time an estimate like (1.6) is proved in the case of nonempty boundary ∂M and a potential V non-identically zero.

2 Uniform a priori estimates

Throughout this section, given a domain $M_0 \subset M$, the Sobolev space $H^1(M_0, d\mathrm{Vol}_g)$ will be equipped with the semi-classical norm defined by

$$\|u\|^2_{H^1(M_0,d\mathrm{Vol}_g)} := \|u\|^2_{L^2(M_0,d\mathrm{Vol}_g)} + \|h\nabla_g u\|^2_{L^2(M_0,d\mathrm{Vol}_g)},$$

where ∇_g denotes the gradient corresponding to Δ_g.

Proposition 2.1. *Under the assumptions of Theorem 1.1, given any $u \in D(G(h))$ and any $a \geq r_0$, the following estimate holds:*

$$\begin{aligned} \|u\|_{H^1(M\setminus X_a,d\mathrm{Vol}_g)} \leq {} & Ch^{-1}\|(h^2\Delta_g + V - E + i\varepsilon)u\|_{L^2(M\setminus X_{a+1},d\mathrm{Vol}_g)} \\ & + C\|u\|_{H^1(X_a\setminus X_{a+1},d\mathrm{Vol}_g)}, \end{aligned} \tag{2.1}$$

for $0 < h \leq h_0$, $0 < \varepsilon \leq 1$, *with constants* $C, h_0 > 0$ *independent of* h *and* ε.

Proof. Let $\eta \in C^\infty(\overline{M})$, $\eta = 1$ in $M \setminus X_a$, $\eta = 0$ in X_{a+1}, and set $w = \eta u \in D(G(h))$. Then (2.1) would follow from the estimate

$$\|w\|_{H^1(M,d\mathrm{Vol}_g)} \leq Ch^{-1}\|(h^2\Delta_g + V - E + i\varepsilon)w\|_{L^2(M,d\mathrm{Vol}_g)}. \tag{2.2}$$

We will derive (2.2) from the following a priori estimate

Proposition 2.2. *Let* $\mathcal{U}(t,x) = 0$ *in* $\mathbf{R} \times X_{a+1}$ *satisfy the equation*

$$\begin{aligned} ((E - V(x))\partial_t^2 + \Delta_g)\mathcal{U}(t,x) &= \mathcal{V}(t,x) \quad \text{in} \quad \mathbf{R} \times M, \\ B\mathcal{U}(t,x) &= 0 \qquad \text{on} \qquad \mathbf{R} \times \partial M. \end{aligned} \tag{2.3}$$

Then, if E *is a non-trapping level, there exist constants* $C, T > 0$ *so that the following inequality holds*

$$\begin{aligned} \|\partial_t \mathcal{U}(T,\cdot)\| + \|\nabla_g \mathcal{U}(T,\cdot)\| \leq C\|\mathcal{U}(0,\cdot)\| + C\|\partial_t \mathcal{U}(0,\cdot)\|_{-2} \\ + C\int_0^T \|\mathcal{V}(t,\cdot)\| dt, \end{aligned} \tag{2.4}$$

where $\|\cdot\|$ *denotes the norm in* $L^2(M, d\mathrm{Vol}_g)$, *while* $\|\cdot\|_{-2}$ *denotes the classical norm in the Sobolev space* $H^{-2}(M, d\mathrm{Vol}_g)$.

Proof. Denote by L_E the self-adjoint realization of the operator $(E - V)^{-1}\Delta_g$ on the Hilbert space $H_E = L^2(M, (E - V)d\mathrm{Vol}_g)$ with boundary conditions $Bu = 0$. By Duhamel's formula we have

$$\begin{aligned} \mathcal{U}(t,\cdot) = \cos\left(t\sqrt{L_E}\right)\mathcal{U}(0,\cdot) + \frac{\sin\left(t\sqrt{L_E}\right)}{\sqrt{L_E}}\partial_t \mathcal{U}(0,\cdot) \\ + \int_0^t \frac{\sin\left((t-\tau)\sqrt{L_E}\right)}{\sqrt{L_E}}\widetilde{\mathcal{V}}(\tau,\cdot)d\tau, \end{aligned} \tag{2.5}$$

where $\widetilde{\mathcal{V}} = (E - V)^{-1}\mathcal{V}$. Let $\chi \in C^\infty(\overline{M})$, $\chi = 1$ on supp $\mathcal{U}$, $\chi = 0$ outside a small neighbourhood of supp $\mathcal{U}$. In view of (2.5) we can write

$$\begin{aligned} \partial_t \mathcal{U}(t,\cdot) &= -L_E \chi \frac{\sin\left(t\sqrt{L_E}\right)}{\sqrt{L_E}} \chi \mathcal{U}(0,\cdot) \\ &+ [L_E, \chi]\frac{\sin\left(t\sqrt{L_E}\right)}{\sqrt{L_E}}\chi \mathcal{U}(0,\cdot) \\ &+ \chi \cos\left(t\sqrt{L_E}\right)\chi\partial_t \mathcal{U}(0,\cdot) + \int_0^t \chi \cos\left((t-\tau)\sqrt{L_E}\right)\chi\widetilde{\mathcal{V}}(\tau,\cdot)d\tau, \end{aligned} \tag{2.6}$$

$$\begin{aligned}\nabla_g \mathcal{U}(t,\cdot) &= \nabla_g \chi \cos\left(t\sqrt{L_E}\right)\chi\,\mathcal{U}(0,\cdot) + \nabla_g \chi \frac{\sin\left(t\sqrt{L_E}\right)}{\sqrt{L_E}}\chi\,\partial_t \mathcal{U}(0,\cdot) \\ &+ \int_0^t \chi \nabla_g \frac{\sin\left((t-\tau)\sqrt{L_E}\right)}{\sqrt{L_E}}\chi\,\widetilde{\mathcal{V}}(\tau,\cdot)d\tau \\ &+ \int_0^t [\nabla_g,\chi] \frac{\sin\left((t-\tau)\sqrt{L_E}\right)}{\sqrt{L_E}}\chi\,\widetilde{\mathcal{V}}(\tau,\cdot)d\tau.\end{aligned} \tag{2.7}$$

It follows from Melrose-Sjöstrand's result on propagation of C^∞ singularities (see [7], [8]) that the distribution kernels of the operators $\chi \cos\left(T\sqrt{L_E}\right)\chi$ and $\chi \frac{\sin\left(T\sqrt{L_E}\right)}{\sqrt{L_E}}\chi$ belong to $C^\infty(\overline{M}\times\overline{M})$ for some $T > 0$ depending on $\operatorname{supp}\chi$. Now (2.4) follows from (2.6), (2.7) and the inequality

$$\|\nabla_g f\|^2_{H_E} \le C\|\nabla_g f\|^2 = C\langle \Delta_g f, f\rangle$$

$$= C\langle L_E f, f\rangle_{H_E} = C\|\sqrt{L_E} f\|^2, \quad \forall f \in D(L_E). \qquad \square$$

Let us apply (2.4) with $\mathcal{U}(t,x) = e^{it/h}w(x)$,

$$\mathcal{V}(t,x) = e^{it/h}h^{-2}(h^2\Delta_g + V - E)w.$$

We get

$$\begin{aligned}\|w\| + \|h\nabla_g w\| \le O(h)\|w\| &+ O(1)\|w\|_{-2} \\ &+ O(h^{-1})\|(h^2\Delta_g + V - E)w\|.\end{aligned} \tag{2.8}$$

On the other hand, we have

$$\begin{aligned}\|w\|_{-2} &\le C\|(E-V)w\|_{-2} \le C\|(h^2\Delta_g + V - E)w\|_{-2} + Ch^2\|\Delta_g w\|_{-2} \\ &\le O(1)\|(h^2\Delta_g + V - E)w\| + O(h^2)\|w\|.\end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9), and taking h small enough lead to the estimate

$$\|w\|_{H^1} \le O(h^{-1})\|(h^2\Delta_g + V - E)w\|. \tag{2.10}$$

On the other hand, by Green's formula we have

$$\|(h^2\Delta_g + V - E + i\varepsilon)w\|^2 = \|(h^2\Delta_g + V - E)w\|^2 + \varepsilon^2\|w\|^2,$$

so

$$\|(h^2\Delta_g + V - E)w\| \le \|(h^2\Delta_g + V - E + i\varepsilon)w\|. \tag{2.11}$$

Now (2.2) follows from (2.10) and (2.11). $\square$

Proposition 2.3. *There exists a constant $b \gg r_0$ so that if $u \in H^2(X_b, d\mathrm{Vol}_g)$, is such that*

$$r^s(h^2\Delta_g + V - E + i\varepsilon)u \in L^2(X_b, d\mathrm{Vol}_g)$$

for $0 < s - 1/2 \ll 1$, $0 < \varepsilon \le 1$, then $\forall 0 < \gamma \ll 1$ there exist constants $C_1, C_2, h_0 > 0$ independent of h and ε (but depending on γ) so that for $0 < h \le h_0$ we have

$$\|r^{-s}u\|^2_{H^1(X_{b+1}, d\mathrm{Vol}_g)} \le C_1 h^{-2} \|r^s(h^2\Delta_g + V - E + i\varepsilon)u\|^2_{L^2(X_b, d\mathrm{Vol}_g)}$$
$$-C_2 h \mathrm{Im}\, \langle \partial_r u, u\rangle_{L^2(\partial X_b)} + \gamma \|u\|^2_{H^1(X_b \setminus X_{b+1}, d\mathrm{Vol}_g)}. \tag{2.12}$$

Remark. This proposition has been proved in [3] (Proposition 2.4) for every $b \ge r_0$ in the case when $q_2 \equiv 0$ and $V \equiv 0$. When the potential V is not identically zero, however, one needs to take the parameter b big enough and $0 < h \le h_0(b) \ll 1$. The proof in this more general case is similar to that in [3], but we will present it below for the sake of completeness.

Let us see that (2.1) and (2.12) imply (1.6). By Green's formula we have

$$\begin{aligned} &- h^2 \mathrm{Im}\, \langle \partial_r u, u\rangle_{L^2(\partial X_b)} = \\ &- \mathrm{Im}\, \langle (h^2\Delta_g + V - E + i\varepsilon)u, u\rangle_{L^2(M\setminus X_b, d\mathrm{Vol}_g)} - \varepsilon \|u\|^2_{L^2(M\setminus X_b, d\mathrm{Vol}_g)} \\ &\le C\gamma_1 h \|\chi_s u\|^2_{L^2(M, d\mathrm{Vol}_g)} \\ &+ C\gamma_1^{-1} h^{-1} \|\chi_{-s}(h^2\Delta_g + V - E + i\varepsilon)u\|^2_{L^2(M, d\mathrm{Vol}_g)}, \end{aligned} \tag{2.13}$$

$\forall \gamma_1 > 0$. Choose $a = b + 3$. Combining (2.1), (2.12) and (2.13), and choosing the parameters γ and γ_1 small enough, we get

$$\|\chi_s u\|_{H^1(M, d\mathrm{Vol}_g)} \le C h^{-1} \|\chi_{-s}(h^2\Delta_g + V - E + i\varepsilon)u\|_{L^2(M, d\mathrm{Vol}_g)},$$
$$\forall u \in D(G(h)), \tag{2.14}$$

for $0 < h \le h_0$ with constants $C, h_0 > 0$ independent of h and ε. Clearly, (2.14) implies (1.6).

Proof of Proposition 2.3. Denote

$$P := p^{1/2}\left(h^2\Delta_g|_X + V - E + i\varepsilon\right) p^{-1/2} = \mathcal{D}_r^2 + L_r + W - E + i\varepsilon,$$

where $\mathcal{D}_r = -ih\partial_r$, $L_r = h^2\Lambda_r$, $W = V + h^2 q$. Note that (1.3) implies

$$-[\partial_r, L_r] \ge \frac{C}{r} L_r, \quad C > 0. \tag{2.15}$$

In what follows $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will denote the norm and the scalar product on $L^2(S)$. Denote by $L^2(X_b)$ and $H^1(X_b)$ the spaces equipped with the norms

$$\|f\|^2_{L^2(X_b)} = \int_b^\infty \|f(r,\cdot)\|^2 dr,$$

$$\|f\|^2_{H^1(X_b)} = \int_b^\infty \left(\|f(r,\cdot)\|^2 + \|\mathcal{D}_r f(r,\cdot)\|^2 + \langle L_r f(r,\cdot), f(r,\cdot)\rangle\right) dr.$$

Choose a function $\phi \in C^\infty(\mathbf{R})$, $0 \le \phi \le 1$, such that $\phi(r) = 0$ for $r \le b + 1/2$, $\phi(r) = 1$ for $r \ge b + 2/3$, and $\phi'(r) \ge 0$, $\forall r$. Set $w = p^{1/2}u$ and

$$F(r) = -\langle (L_r + W_1 - E)\phi w(r,\cdot), \phi w(r,\cdot)\rangle + \|\mathcal{D}_r(\phi w)(r,\cdot)\|^2,$$

where $W_1 = V + h^2 q_1 = W - h^2 q_2$. It is easy to see that the first derivative of $F(r)$ satisfies

$$\begin{aligned}
F'(r) = &- \langle [\partial_r, L_r]\phi w(r,\cdot), \phi w(r,\cdot)\rangle - \langle W_1'\phi w(r,\cdot), \phi w(r,\cdot)\rangle \\
&- 2\varepsilon \mathrm{Im}\,\langle \phi w(r,\cdot), (\phi w)'(r,\cdot)\rangle \\
&- 2h^{-1}\mathrm{Im}\,\langle \phi(P - h^2 q_2) w(r,\cdot), \mathcal{D}_r(\phi w)(r,\cdot)\rangle \\
&- 2h^{-1}\mathrm{Im}\,\langle [P,\phi] w(r,\cdot), \phi \mathcal{D}_r w(r,\cdot)\rangle \\
&- 2h^{-1}\mathrm{Im}\,\langle [P,\phi] w(r,\cdot), [\mathcal{D}_r,\phi] w(r,\cdot)\rangle \\
&\ge -\langle [\partial_r, L_r]\phi w(r,\cdot), \phi w(r,\cdot)\rangle - \langle W_1'\phi w(r,\cdot), \phi w(r,\cdot)\rangle \\
&- \varepsilon h^{-1}\left(\|\phi w(r,\cdot)\|^2 + \|\mathcal{D}_r(\phi w)(r,\cdot)\|^2\right) \\
&- O_\gamma(h^{-2}) r^{2s}\|(P - h^2 q_2) w(r,\cdot)\|^2 \\
&- O(\gamma) r^{-2s}\|\mathcal{D}_r(\phi w)(r,\cdot)\|^2 \\
&- O(h) r^{-2s}\left(\|w(r,\cdot)\|^2 + \|\mathcal{D}_r w(r,\cdot)\|^2\right),
\end{aligned} \tag{2.16}$$

$\forall \gamma > 0$. In view of (1.2) and (1.4), we have

$$W_1'(r,\theta) \le C r^{-1-\delta}, \tag{2.17}$$

with constants $C > 0$, $\delta = \min\{\delta_0, \delta_1\} > 0$. By (2.15), (2.16) and (2.17) we get, for $r \ge b$,

$$\begin{aligned}
F'(r) \ge &\frac{C}{r}\langle L_r \phi w(r,\cdot), \phi w(r,\cdot)\rangle - O(b^{-\sigma}) r^{-2s}\|\phi w(r,\cdot)\|^2 \\
&- \varepsilon h^{-1}\left(\|\phi w(r,\cdot)\|^2 + \|\mathcal{D}_r(\phi w)(r,\cdot)\|^2\right) \\
&- O_\gamma(h^{-2}) r^{2s}\|(P - h^2 q_2) w(r,\cdot)\|^2 \\
&- O(\gamma) r^{-2s}\|\mathcal{D}_r(\phi w)(r,\cdot)\|^2 \\
&- O(h) r^{-2s}\left(\|w(r,\cdot)\|^2 + \|\mathcal{D}_r w(r,\cdot)\|^2\right),
\end{aligned} \tag{2.18}$$

where $\sigma = \delta + 1 - 2s > 0$. Integrating (2.18) from $t \geq b$ to $+\infty$ and using that $L_r \geq 0$, we get

$$\begin{aligned} F(r) \leq & O(b^{-\sigma}) \int_b^{\infty} r^{-2s} \|\phi w(r,\cdot)\|^2 dr + O(\gamma) \int_b^{\infty} r^{-2s} \|\mathcal{D}_r(\phi w)(r,\cdot)\|^2 dr \\ & + \varepsilon h^{-1} \int_b^{\infty} \left(\|\phi w(r,\cdot)\|^2 + \|\mathcal{D}_r(\phi w)(r,\cdot)\|^2 \right) dr \\ & + O_\gamma(h^{-2}) \int_b^{\infty} r^{2s} \|(P - h^2 q_2) w(r,\cdot)\|^2 dr \\ & + O(h) \int_b^{\infty} r^{-2s} \left(\|w(r,\cdot)\|^2 + \|\mathcal{D}_r w(r,\cdot)\|^2 \right) dr. \end{aligned}$$

Hence

$$\begin{aligned} \int_b^{\infty} r^{-2s} F(r) dr \leq & O(b^{-\delta}) \|r^{-s} \phi w\|^2_{L^2(X_b)} \\ & + O(\gamma) \|r^{-s} \mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} \\ & + O(\varepsilon h^{-1}) \left(\|\phi w\|^2_{L^2(X_b)} + \|\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} \right) \\ & + O_\gamma(h^{-2}) \|r^s (P - h^2 q_2) w\|^2_{L^2(X_b)} \\ & + O(h) \|r^{-s} w\|^2_{H^1(X_b)}. \end{aligned} \tag{2.19}$$

On the other hand, multiplying (2.18) by r^{1-2s} and integrating from b to $+\infty$ we get

$$\begin{aligned} (2s-1) & \int_b^{\infty} r^{-2s} F(r) dr = \int_b^{\infty} r^{1-2s} F'(r) dr \\ & \geq C \int_b^{\infty} r^{-2s} \langle L_r(\phi w)(r,\cdot), \phi w(r,\cdot) \rangle dr \\ & - O(b^{-\delta}) \|r^{-s} \phi w\|^2_{L^2(X_b)} - O(\gamma) \|r^{-s} \mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} \\ & - O(\varepsilon h^{-1}) \left(\|\phi w\|^2_{L^2(X_b)} + \|\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} \right) \\ & - O_\gamma(h^{-2}) \|r^s (P - h^2 q_2) w\|^2_{L^2(X_b)} - O(h) \|r^{-s} w\|^2_{H^1(X_b)}. \end{aligned} \tag{2.20}$$

On the other hand, we have

$$\langle (L_r + W - E)\phi w, \phi w \rangle_{L^2(X_b)} + \|\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} = \mathrm{Re}\, \langle P(\phi w), \phi w \rangle_{L^2(X_b)},$$

and hence

$$\begin{aligned}\|\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} &\leq C\|\phi w\|^2_{L^2(X_b)} + \|P(\phi w)\|^2_{L^2(X_b)} \\ &\leq C\|w\|^2_{L^2(X_b)} + \|Pw\|^2_{L^2(X_b)} + O(h^2)\|r^{-s}w\|^2_{H^1(X_b)}.\end{aligned} \tag{2.21}$$

Furthermore we have

$$\begin{aligned}\varepsilon\|w\|^2_{L^2(X_b)} =& \mathrm{Im}\,\langle Pw, w\rangle_{L^2(X_b)} - h^2\mathrm{Im}\,\langle\partial_r w, w\rangle_{L^2(\partial X_b)} \\ &\leq \gamma^{-1}h^{-1}\|r^s Pw\|^2_{L^2(X_b)} + \gamma h\|r^{-s}w\|^2_{L^2(X_b)} \\ &- h^2\mathrm{Im}\,\langle\partial_r w, w\rangle_{L^2(\partial X_b)}.\end{aligned} \tag{2.22}$$

By (2.21) and (2.22),

$$\begin{aligned}\varepsilon h^{-1}\left(\|\phi w\|^2_{L^2(X_b)} + \|\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)}\right) &\leq O_\gamma(h^{-2})\|r^s Pw\|^2_{L^2(X_b)} \\ + O(\gamma)\|r^{-s}w\|^2_{H^1(X_b)} - Ch\mathrm{Im}\,\langle\partial_r w, w\rangle_{L^2(\partial X_b)},\end{aligned} \tag{2.23}$$

$\forall\gamma > 0$, $0 < h \leq h_0(\gamma)$, with a constant $C > 0$. Integrating by parts it is easy to obtain the following estimate:

$$\begin{aligned}&\left|\langle r^{-2s}(L_r + W - E)\phi w, \phi w\rangle_{L^2(X_b)} + \|r^{-s}\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)}\right| \\ &\quad\leq O(h^{-1})\|Pw\|^2_{L^2(X_b)} + O(h)\|r^{-s}w\|^2_{H^1(X_b)}.\end{aligned} \tag{2.24}$$

Since $E - W \geq E - V - O(h) \geq C_2 - O(h) \geq C_2/2 > 0$, we deduce from (2.24),

$$\begin{aligned}\|r^{-s}\phi w\|^2_{L^2(X_b)} \leq& C\|r^{-s}\mathcal{D}_r(\phi w)\|^2_{L^2(X_b)} + C\langle r^{-2s}L_r(\phi w), \phi w\rangle_{L^2(X_b)} \\ &+ O(h^{-1})\|Pw\|^2_{L^2(X_b)} + O(h)\|r^{-s}w\|^2_{H^1(X_b)}.\end{aligned} \tag{2.25}$$

Combining (2.19), (2.20), (2.23), (2.24) and (2.25), we get

$$\begin{aligned}\|r^{-s}\phi w\|^2_{H^1(X_b)} &\leq O(b^{-\delta})\|r^{-s}\phi w\|^2_{L^2(X_b)} + O(\gamma)\|r^{-s}w\|^2_{H^1(X_b)} \\ &- Ch\mathrm{Im}\,\langle\partial_r w, w\rangle_{L^2(\partial X_b)} + O_\gamma(h^{-2})\|r^s Pw\|^2_{L^2(X_b)},\end{aligned} \tag{2.26}$$

$\forall\gamma > 0$, $0 < h \leq h_0(\gamma)$, with a constant $C > 0$, provided $s - 1/2 > 0$ is small enough. Clearly, (2.12) follows from (2.26) by taking b big enough, $\gamma > 0$ small enough depending on b, and $0 < h \leq h_0(b, \gamma) \ll 1$. □

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